FRW Universe Models in Conformally Flat Spacetime Coordinates

II: Universe models with negative and vanishing spatial curvature

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Abstract We deduce general expressions for the line element of universe models with negative and vanishing spatial curvature described by conformally flat spacetime coordinates. The empty Milne universe model and models with dust, radiation and vacuum energy are exhibited. Discussing the existence of particle horizons we show that there is continual creation of space,matter and energy when conformal time is used in Friedmann-Robertson-Walker models with negative spatial curvature.

1. Introduction.

In the first paper [1] of this series we have developed a general formalism for describing Friedmann-Robertson-Walker (FRW) universe models using conformally flat spacetime coordinates (CFS) [2-9]. The line element is written as a conformal factor times the Minkowski line element,

$$ds^2 = A(T,R)^2(-dT^2 + dR^2 + R^2 d\Omega^2) \ , \tag{1}$$

in such coordinates. Using standard cosmic coordinates the line element has the form

$$ds^2 = -dt^2 + a(t)^2[\,d\chi^2 + S_k(\chi)^2 d\Omega^2] \ , \tag{2}$$

where

$$S_k(\chi) = \begin{cases} \sin\chi & \text{for} \quad k=1 \quad , \quad 0<\chi<\pi \\ \chi & \text{for} \quad k=0 \quad , \quad 0<\chi<\infty \\ \sinh\chi & \text{for} \quad k=-1 \quad , \quad 0<\chi<\infty \end{cases} \ . \tag{3}$$

Introducing parametric time

$$\eta = \int_{t_1}^{t} \frac{dt}{a(t)} \tag{4}$$

where t_1 is an arbitrary constant, the line element takes the form

$$ds^2 = a(\eta)^2\,[\,-d\eta^2 + d\chi^2 + S_k(\chi)^2 d\Omega^2] \ . \tag{5}$$

In reference [1] it is shown that the transformation

$$T = \frac{1}{2}\,[\,f(\eta+\chi) + f(\eta-\chi)] \quad , \quad R = \frac{1}{2}\,[\,f(\eta+\chi) - f(\eta-\chi)] \tag{6}$$

with

$$f(x) = c\left[b + I_k\left(\tfrac{x-a}{2}\right)\right]^{-1} + d \ , \tag{7}$$

where a, b, c, d are arbitrary constants and

$$I_k(x) = \begin{cases} \cot x & \text{for} \quad k=1 \\ 1/x & \text{for} \quad k=0 \\ \coth x & \text{for} \quad k=-1 \end{cases} \quad , \tag{8}$$

leads from (5) to (1) with

$$A(T,R) = \tfrac{a(\eta(T,R))\, S_k(\chi(T,R))}{|R|} \ . \tag{9}$$

We shall require that the CFS time coordinate T is an increasing function of the cosmic time t, and hence of the parametric time η, for every value of χ. Then it follows from equation (6) that f must be an increasing function. Hence equation (54) in reference [1] implies that $c > 0$.

In the present paper this formalism shall be applied to FRW universe models with $k = -1$ and $k = 0$.

2. CFS coordinates in negatively curved universe models

M.J.Chodorowski [10] has recently presented an interesting discussion of the concept *space* in a cosmological context. He has deduced the form of the Robertson-Walker line element for the case of a negatively curved space as expressed in terms of conformal time, and pointed out that space is a coordinate dependent concept.

In this case $S_{-1}(\chi) = \sinh\chi$. The CFS coordinates used by Chodorowski may be obtained by choosing the values $a = 0$, $b = -1$, $c = 2T_i$ and $d = T_i$ in equation (7), where T_i is the conformal time corresponding to $t = \eta = 0$ at $\chi = 0$. This gives the generating function

$$f(x) = T_i e^x \ . \tag{10}$$

Hence the transformation (6) between the coordinates (η, χ) and the conformal coordinates (T, R) is

$$T = T_i e^\eta \cosh\chi \quad , \quad R = T_i e^\eta \sinh\chi \ , \tag{11}$$

The inverse transformation is

$$T_i e^\eta = \sqrt{T^2 - R^2} \quad , \quad \tanh\chi = \frac{R}{T} \ , \tag{12}$$

where $R > 0$ and $T^2 > R^2$. At $\chi = R = 0$, the clocks showing conformal time goe exponentially faster than the clocks showing parametric time. Equations (3), (11) and (12) lead to

$$S_{-1}(\chi(T,R)) = \tfrac{R^2}{T^2 - R^2} \ . \tag{13}$$

It follows from equations (13) and (9) that the line element for a universe model with negative spatial curvature, as expressed in terms of the conformal coordinates (T, R), takes the conformally flat form

$$ds^2 = \frac{a(\eta(T,R))^2}{T^2 - R^2}\, ds_M^2 \ . \tag{14}$$

In the (T, R)-system, each reference particle with $\chi =$ constant in the cosmic coordinate system has a constant recession velocity

$$V = \frac{R}{T} = \tanh \chi \tag{15}$$

which is less than 1. According to this equation χ is the rapidity of a reference particle with radial coordinate χ. The question of superluminal expansion of space using the line element (14) has been discussed by Lewis et al.[11].

Figure 1 shows the cosmic coordinate system (η, χ) in a Minkowski diagram referring to the conformal coordinate system of the observer at $\chi = 0$. It follows from equations (12) that the world lines of the reference particles with $\chi =$ constant are straight lines, and the curves of the cosmic space $\eta =$ constant are hyperbolae with centre at the origin as show in the diagram in Figure 1.

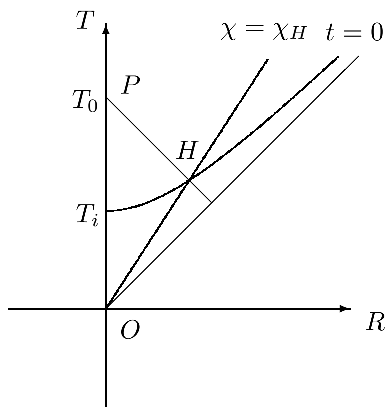

Figure 1. Minkowski diagram for universe models with negative spatial curvature with reference to the conformal coordinates (T, R). Here the line OH is the world line of a reference particle with $\chi =$ constant. The line PH represents the backwards light cone. The hyperbola represents the cosmic space at cosmic time $t = 0$.

The hyperbola shown in Figure 1 represents the Big Bang singularity of the universe, i.e. the spacetime boundary of the universe. Conformal space is represented by a horizontal line in this figure from the time axis to the hyperbola. Hence the conformal space of the universe has finite extension even if the cosmic space has infinite extension.

Equation (11) shows that the velocity of the (T, R)-system relative to the (η, χ)-system, i.e. of a particle with $R =$ constant, is given by

$$\frac{d\chi}{d\eta} = -\tanh \chi \ . \tag{16}$$

Hence the (T, R)-system contracts relative to the (η, χ)-system, i.e. relative to the Hubble flow. In a (η, χ)-diagram the world lines of the reference particles with $R = R_1$ are given by

$$\sinh \chi = \frac{R_1}{T_i} e^{-\eta} \ , \tag{17}$$

and the simultaneity curves $T = T_1$ of the conformal space by

$$\cosh\chi = \frac{T_1}{T_i}\, e^{-\eta} \; . \tag{18}$$

These curves are shown in Figure 2.

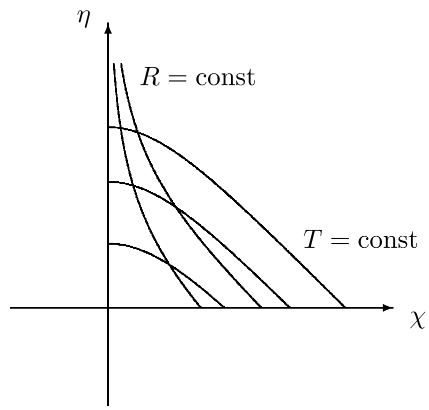

Figure 2. Minkowski diagram for universe models with negative spatial curvature with reference to the (η, χ)-coordinate system. The diagram shows world lines $R =$ constant and simultaneity curves $T =$ constant. We see that the (T, R)-system contracts relative to the (η, χ)-system, i.e. relative to the Hubble flow.

3. The Milne universe model

The Milne universe is a model of an empty universe with $\Omega_0 = 0$. This model has negative spatial curvature. The scale factor is

$$a(t) = t \; . \tag{19}$$

The normalisation condition $a(t_0) = 1$ is fullfilled since the unit of time is chosen to be equal to the present age t_0 of the Milne universe. Equation (4) then gives

$$e^{\eta} = t \tag{20}$$

with $\eta \in \, <\!-\infty, \infty\!>$ when $t \in \, <\!0, \infty\!>$.

For the Milne universe the transformation between the cosmic coordinates and the conformal coordinates is

$$T = t\cosh\chi \quad , \quad R = t\sinh\chi \; , \tag{21}$$

with inverse transformation

$$t = \sqrt{T^2 - R^2} \quad , \quad \tanh\chi = \frac{R}{T} \; . \tag{22}$$

In this case $T = t$ at $\chi = 0$, i.e. the conformal clock at $\chi = 0$ goes at the same rate as the cosmic clocks. However, while the cosmic clocks go at the same rate at all positions,

the conformal clocks go faster for larger values of χ. Inserting equation (22) into equation (14) gives

$$ds^2 = ds_M^2 \ , \tag{23}$$

which shows that in the case of the Milne universe model the conformal coordinates are the same as the coordinates of flat spacetime in a static reference frame in which the line element takes the Minkowski form. Hence the Hubble parameter H_R in the conformal coordinate system vanishes in this case.

According to equation (15) the conformal coordinate system is comoving with an observer at $\chi = 0$. In the Milne universe the conformal space is then identical to the private space of this observer. Furthermore the public space at $t = 0$ is not represented by a hyperbola in Figure 1, but by the light cone $R = T$. The private space still has a finite extension at the time T. However, the public space at $t = t_1$ is represented by the hyperbola $T^2 - R^2 = t_1^2$ in the (T, R)-diagram, and hence has an infinite extension.

4. Particle horizon in negatively curved universe models using conformal time.

We here consider universe models with negative spatial curvature, and where the parametric time $\eta \to 0$ when the cosmic time $t \to 0$. As noted above, the universe as described in conformally flat coordinates extends only out to the hyperbola in Figure 1, and not out to the light cone.

As an interesting application of the conformal coordinates, we will discuss the existence of a particle horizon using these coordinates. In Figure 1 we have drawn a Minkowski diagram where the hyperbola represents the simultaneity space at constant cosmic time $t = 0$ measured by clocks moving along the straight world lines from the origin.

The standard radial coordinate of the particle horizon is defined by

$$\chi_H = \int_{t_i}^{t} \frac{dt}{a(t)} \ . \tag{24}$$

From equations (24) and (4) it follows that

$$\chi_H = \eta \ . \tag{25}$$

We shall now show how this equation can be deduced directly from the figure.

Consider an observer at the point P having coordinates $(0, T_0)$. The line PH given by $R = T_0 - T$ represents the backwards light cone of this observer. A straight line $\chi =$ constant through the origin represents a spherical surface in space at different times. The particle horizon of this observer is the spherical surface within which he may receive information emitted after cosmic time $t = 0$. The space at $t = 0$ is represented by the hyperbola given by $T^2 - R^2 = T_i^2$. The position of the horizon is given by the intersection between PH and the hyperbola which has coordinates

$$T_H = \frac{T_0^2 + T_i^2}{2T_0} \quad , \quad R_H = \frac{T_0^2 - T_i^2}{2T_0} \ . \tag{26}$$

The time coordinate η_P of the observer is found by inserting $R = 0$ in the first of the equations (12) giving

$$e^{\eta_P} = \frac{T_0}{T_i} \ . \tag{27}$$

The radial coordinate of the horizon χ_H is found from the transformation (11) which leads to

$$e^{\chi_H} = \cosh \chi_H + \sinh \chi_H = \frac{T+R}{T_i e^{\eta_H}} = \frac{T+R}{T_i} \ , \tag{28}$$

since the time coordinate $\eta_H = 0$ at the point H in Figure 1. Inserting the coordinates of the point H from equation (26) into this equation, we arrive at

$$e^{\chi_H} = \frac{T_0}{T_i} \ . \tag{29}$$

Combining equations (27) and (29), we obtain equation (25).

A comoving object in the universe has a position given by a fixed coordinate χ. In the present case this corresponds to a straight worldline in Figure 1. The physical significance of the coordinate χ_H is that an observer at $R = 0$ cannot observe objects with $\chi > \chi_H$ at the point of time T_0.

5. Continual creation in negatively curved universe models with dust and radiation using conformal time.

The scale factor of a universe model with dust and radiation and with negative spatial curvature, is given parametrically by [12,13]

$$a = \alpha(\cosh \eta - 1) + \beta \sinh \eta \tag{30}$$

and

$$t = \alpha(\sinh \eta - \eta) + \beta(\cosh \eta - 1) \ , \tag{31}$$

where

$$\alpha = \frac{\Omega_{m0}}{2(1-\Omega_0)} \quad \text{and} \quad \beta = \sqrt{\frac{\Omega_{\gamma 0}}{1-\Omega_0}} \ . \tag{32}$$

Here Ω_{m0} and $\Omega_{\gamma 0}$ are the present values of the density parameters of dust and radiation, respectively, and $\Omega_0 = \Omega_{m0} + \Omega_{\gamma 0}$. We have that $\eta \in <0, \infty>$ when $t \in <0, \infty>$. Transforming to conformal coordinates by means of equation (11) and choosing

$$T_i = \tfrac{1}{2}(\alpha + \beta) \ , \tag{33}$$

the line element (14) takes the form

$$ds^2 = \left(1 - \frac{\alpha+\beta}{2\sqrt{T^2 - R^2}}\right)^2 \left(1 - \frac{\alpha-\beta}{2\sqrt{T^2 - R^2}}\right)^2 ds_M^2 \ , \tag{34}$$

where $T^2 - R^2 > T_i^2$. Note that when $\beta = 0$, we obtain the line element of a dust dominated universe

$$ds^2 = \left(1 - \frac{\alpha}{2\sqrt{T^2 - R^2}}\right)^4 ds_M^2 \tag{35}$$

in accordance with [6]. Putting $\alpha = 0$ we obtain the line element of a radiation dominated universe with negative spatial curvature in conformal coordinates

$$ds^2 = \left(1 - \frac{\beta^2}{4(T^2 - R^2)}\right)^2 ds_M^2 \ . \tag{36}$$

The relationship between the cosmic time and the conformal time at $R = \chi = 0$ is

$$dt = A(T, 0)dT \ . \tag{37}$$

From equation (34) it then follows that

$$dt = \left(1 - \frac{\alpha+\beta}{2T}\right)\left(1 - \frac{\alpha-\beta}{2T}\right) dT \ , \tag{38}$$

which shows that $dT/dt > 1$, i.e. the clocks showing conformal time go at a faster rate than those showing cosmic time. The expression (38) may be integrated with an integration constant determined by the condition $t = 0$ for $T = T_i$. The result is, however, already contained in the transformation equation (6). With $R = 0$ we have from equation (12)

$$T = T_i e^{\eta} \ . \tag{39}$$

Then, using equation (31) we obtain

$$t = T - \alpha \ln \frac{T}{T_i} - \frac{1}{2}(\alpha - \beta)\frac{T_i}{T} - \beta \ , \tag{40}$$

showing that $t = 0$ for $T = T_i = (\alpha + \beta)/2$.

In a universe model dominated by dust equation (40) reduces to

$$t = \frac{T^2 - T_i^2}{T} - 2T_i \ln \frac{T}{T_i} \ , \tag{41}$$

and in a radiation dominated universe model,

$$t = \frac{(T - T_i)^2}{T} < T \ . \tag{42}$$

Hence the conformal age of the universe at $R = 0$ is larger than the cosmic age, which is a coordinate effect.

An interesting phenomenon taking place in the case where the universe extends only out to the hyperbola in Figure 1, is continual creation of new space, matter and radiation (Figure 3).

At the local time T_1 the whole of the universe is represented by the horizontal line segment P_1Q_1. At the time T_2 this space has expanded so that it is now represented by the line segment P_2Q_2. This space may be identified by reference particles enumerated by χ with $0 < \chi < \chi_1$. The part of the line $T = T_2$ outside P_2Q_2 and inside the hyperbola represents new space which has been created after the time T_1. This space contains new reference particles with $\chi > \chi_1$ that did not exist at the point of time $T = T_1$. The enlargement of conformal space is therefore due partly to expansion and partly to creation of new space, matter and radiation.

The usual picture of Big Bang using cosmic time t and standard radial coordinate χ is that it happened everywhere at a certain point of time $t = 0$. The conformally flat picture is radically different. The hyperbola $t = 0$ in figures 3a and 3b represents the creation of the universe. In the conformally flat coordinate system this creation appears as a spherical front expanding from the position $R = 0$ of the observer. According to the

conformally flat picture of the universe an arbitrary observer finds that the universe is isotropic, but not homogeneous. He also finds that he is positioned at its center. Particles and radiation are emitted at a spherical front with infinitely high energy density and temperature, representing the boundary of the universe, moving radially outwards with superluminal velocity dR/dT. Hence there is continual creation of space, matter and energy in the universe as depicted in conformally flat coordinates.

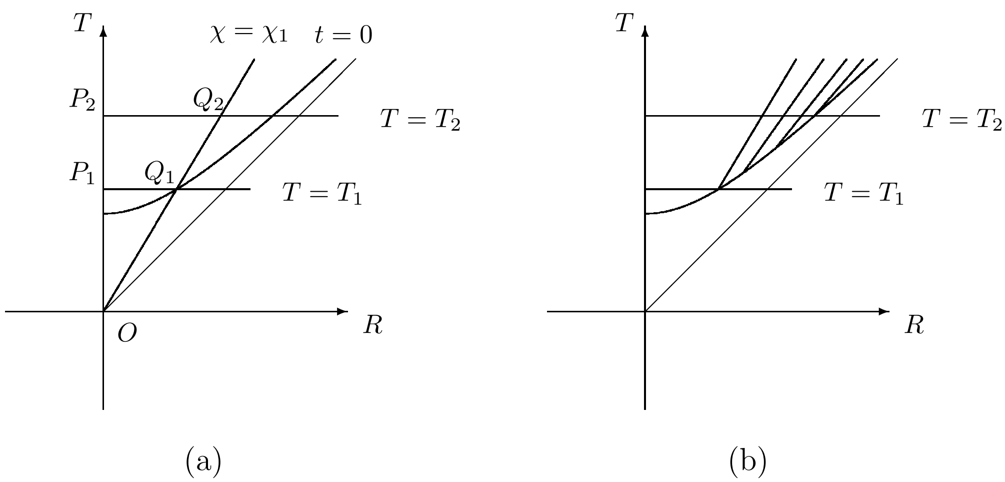

Figure 3. Minkowski diagram of a universe model with radiation and dust using conformal coordinates of type I. The horizontal line segment P_1Q_1 in (a) represents the conformal space at the conformal time T_1. At the conformal time $T = T_2$ the space P_1Q_1 that existed at $T = T_1$ has expanded to P_2Q_2. The figures show that in addition, new conformal space has been created on the line $T = T_2$ outside the line segment P_2Q_2 and inside the hyperbola, as indicated by the world lines in (b).

Although the conformal space is finite, a space traveller can never reach the boundary of the universe since it moves outwards with a velocity greater than the velocity of light. Hence no observer will ever risk arriving at this terrible inferno.

Differentiation of equation (12) while keeping $\eta = 0$ gives the the following expression for the velocity of the front

$$\left(\frac{dR}{dT}\right)_{\eta=0} = \frac{1}{\sqrt{1-T_i^2/T^2}} \tag{43}$$

at an arbitrary time $T > T_i$. On the other hand, the expansion of the conformal space represented by the velocity of the Hubble, flow, is

$$\left(\frac{dR}{dT}\right)_{\chi=\chi_1} = \tanh \chi_1 = \sqrt{1 - T_i^2/T_1^2} \; . \tag{44}$$

Note that $(dR/dT)_{\eta=0} > 1$ and $(dR/dT)_{\chi=\chi_1} < 1$. This means that the boundary of the universe moves faster than the inertial flow representing the expansion of space. The initial velocity of the boundary of the universe is infinitely great and then decreases and approaches the velocity of light in an infinitely remote future. On the other hand the conformally flat Hubble flow has a time independent velocity less than that of light.

In the conformally flat picture new space is continually created at the boundary.

However, using cosmic time no new space is created. This apparent contradiction is solved by noting that constant cosmic time and constant local time represent different simultaneities. So the conformal space defined at constant conformal time is different from the cosmic space defined at constant cosmic time.

An exception among the negatively curved universe models is the Milne universe. Due to the special form of the scale factor in this universe model, the parametric time $\eta \to -\infty$ when the cosmic time $t \to 0$. Hence the space $t = 0$ is not represented by a hyperbola, but by the light cone in Figure 1. This means that there is no new space created using local time in this universe model.

The Kretschmann curvature scalar for the models with dust and with radiation are respectively

$$K_d = \frac{245760\,(T^2 - R^2)^3 \alpha^2}{[2\sqrt{T^2 - R^2} - \alpha]^{12}} \ , \quad K_r = \frac{1572864\,(T^2 - R^2)^4 \beta^4}{[4(T^2 - R^2) - \beta^2]^8} \ . \tag{45}$$

These expressions show that there is a physical singularity with infinitely great spacetime curvature at the boundary $T^2 - R^2 = T_i^2$ with continual creation.

6. Negatively curved universe models with vacuum energy

The inflationary era is a brief period dominated by vacuum energy with accelerated expansion at the beginning of the universe. This era is often said to make space flat. That cannot, however, be the case. The inflationary era cannot change a universe model with curved space, $k \neq 0$, to a model with flat space, $k = 0$. It can only make space *approximately* flat. The curvature decreases exponentially. Such a space is still curved, although the curvature may be so small that we are not able to measure it. So if our universe entered the inflationary era with curved space, the space will still be curved today.

The region of initial conditions for universe models with curved space is much larger than that for flat space. Hence, if the universe entered the inflationary era by some sort of quantum processes, the probability that the space is curved is much larger than that it is flat. One may therefore conclude that we probably live in a universe with curved space, but that the space was inflated so much in the inflationary era that we are not able to measure the curvature.

Although the difference between such a curved universe and a flat one is negligible small as far as observed properties of the universe is concerned, there are some very interesting conceptual differences between curved universe models and flat ones. For example a flat universe dominated by Lorentz Invariant Vacuum Energy (LIVE) has a steady state character and may be infinitely old, while a corresponding universe with negatively curved space is evolving and has a finite age. In this case the scale factor is

$$a(t) = \frac{1}{\widehat{H}_\Lambda} \sinh(\widehat{H}_\Lambda t) \ , \tag{46}$$

for $t > 0$, where

$$\widehat{H}_\Lambda = l_0 \sqrt{\Lambda/3} \ , \quad l_0 = \frac{1}{H_0} \sqrt{\frac{1}{1-\Omega_0}} \ , \tag{47}$$

and Λ is the cosmological constant. The dimensionless Hubble parameter, $\widehat{H} = l_0 \dot{a}/a$, is

$$\widehat{H} = \widehat{H}_\Lambda \coth(\widehat{H}_\Lambda t) \ . \tag{48}$$

Using equation (4), the parametric time η is [14]

$$\eta = -\text{arccoth}(\cosh(\widehat{H}_\Lambda t)) = \ln(\tanh \frac{\widehat{H}_\Lambda t}{2}) \ . \tag{49}$$

Note that $\eta \to -\infty$ when $t \to 0$, and that $\eta \to 0$ when $t \to \infty$. This means that the conformal space extends out to the light cone $R = T$. Hence, there is no continual creation in such a universe model. For this universe model the limit $t \to \infty$ at $\chi = 0$ corresponds to a conformal time $T = T_i$ according to equation (23). It is therefore natural in this case to replace the initial time T_i with a final time T_f. Thus the conformal time at $R = 0$ is

$$T = T_f e^\eta = T_f \tanh \frac{\widehat{H}_\Lambda t}{2} < T_f \ , \tag{50}$$

where T_f is the final conformal time. It follows that the LIVE dominated universe with $k = -1$ has a finite conformal age. This is a coordinate effect since

$$dT = \frac{\widehat{H}_\Lambda T_f/2}{\cosh^2\left(\widehat{H}_\Lambda t/2\right)} dt \ , \tag{51}$$

which shows that for $\widehat{H}_\Lambda t >> 1$ the rate of the conformal time decreases exponentially compared to the rate of the cosmic time.

This model is, however, not realistic since the general theory of relativity is valid only after the Planck time, $t_{Pl} = 5.4 \cdot 10^{-44} s$. Before this time the universe may have existed in a quantum era which cannot be described without a quantum theory of gravity. We here assume that the universe entered a vacuum dominated inflationary era at the Planck time. This implies that the conformal space only extends out to a hyperbola given by $t = t_{Pl}$, which represents the frontier against a quantum era with properties that we cannot describe with the present theories. At this frontier there is continual creation of new conformal space.

Equations (46) and (49) imply that the scale factor is given in terms of the parametric time as

$$a(\eta) = -\frac{1}{\widehat{H}_\Lambda \sinh \eta} \ . \tag{52}$$

Together with equation (14) this relation implies that that the line element for the present universe model, as expressed by conformal coordinates, takes the form

$$ds^2 = \frac{1}{\Omega_0 \widehat{H}_0^2} \left[\frac{2T_f}{T_f^2 - (T^2 - R^2)}\right]^2 ds_M^2 \ . \tag{53}$$

This expression shows that the relationship between cosmic time and conformal time is

$$dt = \frac{1}{\widehat{H}_0 \sqrt{\Omega_0}} \ \frac{2T_f}{T_f^2 - (T^2 - R^2)} \ dT \ . \tag{54}$$

The coordinate t is the proper time of the freely moving observers. The expression shows that the clocks showing conformal time slows down towards a vanishing rate of time as we approach the boundary $T^2 - R^2 = T_f^2$ of the conformal space.

The conformal space has an interesting behaviour in this universe model. Since the parametric time $\eta = 0$ corresponds to cosmic time $t \to \infty$, the hyperbola $\eta = 0$ in Figure 4 represents a future limit beyond which there exists no space. Hence the figure shows that conformal space is annihilated in the LIVE dominated universe. At the point of time $T = T_f$, conformal space starts vanishing at $R = 0$. Then a spherical hole develops which does not belong to the conformal space. Note that space consists of simultaneous events in spacetime, and the points inside the hole do not correspond to events in the FRW-universe. An observer can never reach the boundary of the hole since the cosmic time approaches infinity at this boundary. At the local time T_1 the hole in the universe is represented by the horizontal line segment P_1Q_1. At the time T_2 this hole has expanded so that it is now represented by the line segment P_2Q_2. The part of the line $T = T_2$ outside P_2Q_2 and inside the hyperbola represents new emptiness which has appeared after the time T_1. The enlargement of the hole in conformal space is therefore due partly to expansion and partly to annihilation of space.

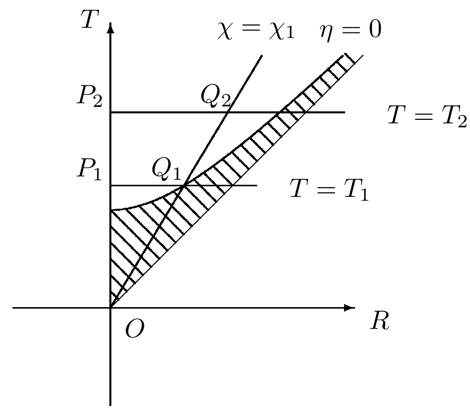

Figure 4. The hatched region represents a succession of conformal spaces in a LIVE dominated universe at different points of time. The hyperbola $\eta = 0$ corresponds to cosmic time $t \to \infty$. The horizontal line segment P_1Q_1 represents a hole in the conformal space at the conformal time T_1. At the conformal time $T = T_2$ the hole P_1Q_1 that existed at $T = T_1$ has expanded to P_2Q_2. In addition the figure shows that conformal space has been annihilated on the line $T = T_2$ outside the line segment P_2Q_2 and inside the hyperbola.

At the present time the universe is filled with radiation, matter and vacuum energy. If the vacuum energy is of the LIVE type, the energy density will remain constant in the future. But the density of radiation and matter will decrease. Hence the universe will approach a vacuum dominated state. This means that in conformal coordinates the final destiny of our universe will be as described above.

7. A second type of conformal coordinates for universe models with negative spatial curvature

For universe models with negative spatial curvature one may introduce a second type of conformal coordinates $(\widehat{T}, \widehat{R})$ by choosing $a = 0$, $b = 0$, $c = 1$ and $d = 0$ in equation (7).

This gives the generating function

$$f(x) = \tanh(x/2) \ . \tag{55}$$

The transformation (6) then takes the form

$$\widehat{T} = \frac{\sinh\eta}{\cosh\eta + \cosh\chi} \quad , \quad \widehat{R} = \frac{\sinh\chi}{\cosh\eta + \cosh\chi} \ . \tag{56}$$

If the universe model begins at $\eta = 0$, this transformation maps the first quadrant $\chi > 0$, $\eta > 0$ onto the triangle $0 < \widehat{T} < 1$ and $0 < \widehat{R} < 1 - \widehat{T}$. On the other hand, if the universe is infinitely old, the fourth quadrant $\chi > 0$, $\eta < 0$ is mapped onto the triangle $-1 < \widehat{T} < 0$ and $0 < \widehat{R} < 1 + \widehat{T}$. The inverse transformation is

$$\coth\eta = \frac{1+\left(\widehat{T}^2-\widehat{R}^2\right)}{2\widehat{T}} \quad , \quad \coth\chi = \frac{1-\left(\widehat{T}^2-\widehat{R}^2\right)}{2\widehat{R}} \ . \tag{57}$$

From equation (9) it then follows that the line element now takes the form

$$ds^2 = \frac{4\,a(\eta(\widehat{T},\widehat{R}))^2}{[1-(\widehat{T}^2-\widehat{R}^2)]^2 - 4\widehat{R}^2} ds_M^2 \ . \tag{58}$$

As drawn in the $(\widehat{T},\widehat{R})$ spacetime diagrams in Figures 5 and 6, both the world lines of points with $\chi = \chi_0$ and the simultaneity curves $\eta = \eta_0$ are hyperbolae, given respectively by

$$(\widehat{R} - a_1)^2 - \widehat{T}^2 = a_1^2 - 1 \quad \text{and} \quad (\widehat{T} - b_1)^2 - \widehat{R}^2 = b_1^2 - 1 \ , \tag{59}$$

where $a_1 = \coth\chi_0$ and $b_1 = \coth\eta_0$. These equations are valid both for universe models dominated by radiation and dust and by LIVE, the only difference being that $\eta_0 > 0$ with radiation and dust, and $\eta_0 < 0$ with LIVE. The velocity in the $(\widehat{T},\widehat{R})$-system of a particle with $\chi =$ constant is given by

$$V = \left(\frac{d\widehat{R}}{d\widehat{T}}\right)_{\chi=\text{constant}} = -\frac{\sinh\eta\sinh\chi}{1+\cosh\eta\cosh\chi} = \frac{2\widehat{T}\widehat{R}}{-1+(\widehat{T}^2+\widehat{R}^2)} \ . \tag{60}$$

In this coordinate system the initial recession velocity vanishes. Note that $V < 0$ for $\widehat{T}\widehat{R} > 0$ and $V > 0$ for $\widehat{T}\widehat{R} < 0$ since $\widehat{T}^2 + \widehat{R}^2 < 1$.

The reference particles of the $(\widehat{T},\widehat{R})$-system and the (T,R)-system have different motions. The velocity in the (η,χ)-system of a particle with $\widehat{R} =$ constant is

$$\frac{d\chi}{d\eta} = \frac{\sinh\eta\sinh\chi}{1+\cosh\eta\cosh\chi} \ . \tag{61}$$

Hence the $(\widehat{T},\widehat{R})$-system expands relative to the (η,χ)-system while the (T,R)-system contracts for $\widehat{T} > 0$.

World lines of free particles defining the inertial Hubble flow and simultaneity curves $\eta =$ constant in a universe with negative spatial curvature containing dust and radiation are depicted relative to the $(\widehat{T},\widehat{R})$-system in Figure 5. This figure gives an illusion of a contracting universe of finite extension with a finite age, since the world lines of

particles with χ = constant, which define the Hubble flow, all terminate at the point $(\widehat{T}, \widehat{R}) = (1, 0)$. However, according to equation (57), approaching this point means that the parametic time η, and hence also the cosmic time t and the scale factor, approaches infinity. Furthemore, this equation also implies that χ approaches infinity at the line $\widehat{T} = 1 - \widehat{R}$.

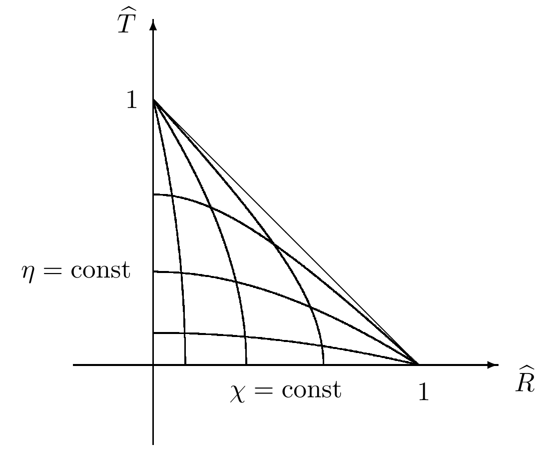

Figure 5. Minkowski diagram for universe models with negative spatial curvature containing dust and radiation with reference to the conformal coordinate system of type II defined in equation (56). The diagram shows world lines χ = constant and simultaneity curves η = constant. We see that the (η, χ)-system contracts relative to the $(\widehat{T}, \widehat{R})$-system. The line $\widehat{T} + \widehat{R} = 1$ represents the limit $\eta \to \infty$, $\chi \to \infty$.

Figure 6 shows a corresponding Minkowski diagram as that of Figure 5, but this time for a universe filled with vacuum energy. This figure provides the picture of a universe of finite extension with a finite age, and with attractive gravitation. Since the world lines of the particles with χ = constant curve towards the $\widehat{T}$-axis, the coordinate velocity of these particles decelerates. According to equations (49) and (57), the conformal time $\widehat{T} = -1$ when the cosmic time $t = 0$, and $\widehat{T} \to 0$ when $t \to \infty$, meaning that this universe will become infinitely old. From equation (57) it also follows that χ approaches infinity at the line $\widehat{T} = \widehat{R} - 1$. Furthermore, equation (46) implies that $\ddot{a} > 0$, which means that the Hubble flow has accelerated expansion.

An important difference between the CFS systems of type I and II is that in the (T, R)-system the free particles defining the Hubble flow move with constant velocity along straight world lines, while in the $(\widehat{T}, \widehat{R})$-system their world lines are hyperbolae.

For a negatively curved universe with radiation and dust the line element with type II conformal coordinates takes the form

$$ds^2 = \left[\frac{2\alpha\left(1+\widehat{T}^2-\widehat{R}^2-\sqrt{(1+\widehat{T}^2-\widehat{R}^2)^2-4\widehat{T}^2}\right)+4\beta\widehat{T}}{(1+\widehat{T}^2-\widehat{R}^2)^2-4\widehat{T}^2} \right]^2 ds_M^2 \ . \tag{62}$$

The type II conformal time at $\widehat{R} = 0$ is related to the cosmic time by

$$t = \alpha \left(\frac{2\widehat{T}}{1-\widehat{T}^2} - \arcsin \frac{2\widehat{T}}{1-\widehat{T}^2} \right) + \beta \frac{2\widehat{T}^2}{1-\widehat{T}^2} \ . \tag{63}$$

The corresponding relationship for a LIVE dominated universe is

$$\widehat{H}_\Lambda t = \ln \widehat{T} \ . \tag{64}$$

The $\widehat{T}$-clocks goe faster than the cosmic clocks in this universe model.

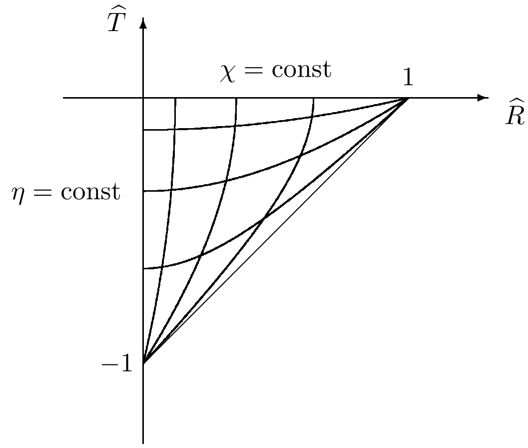

Figure 6. Minkowski diagram for universe models with negative spatial curvature containing vacuum energy with reference to the conformal coordinate system of type II defined in equation (56). The diagram shows world lines $\chi =$ constant and simultaneity curves $\eta =$ constant. We see that in this case the (η, χ)-system expands relative to the $(\widehat{T}, \widehat{R})$-system. The line $\widehat{T} + 1 = \widehat{R}$ represents the limit $\eta \to 0$, $\chi \to \infty$, while $\chi = 0$ on the $\widehat{T}$-axis, and $\eta \to \infty$ on the $\widehat{R}$-axis.

From equations (49), (57) and (58) it follows that with the type II conformal coordinates, the line element of a negatively curved universe model dominated by LIVE takes the form

$$ds^2 = \frac{1}{\widehat{H}_\Lambda^2 \widehat{T}^2} ds_M^2 \ . \tag{65}$$

The coordinate transformation between the present conformal coordinates and those introduced in equation (11) is

$$\widehat{T} = \frac{(T^2 - T_i^2) - R^2}{(T + T_i)^2 - R^2} \quad , \quad \widehat{R} = \frac{2T_i R}{(T + T_i)^2 - R^2} \ . \tag{66}$$

8. A third type of conformal coordinates for universe models with negative spatial curvature

For universe models with negative spatial curvature one may also introduce a third type of conformal coordinates $(\widetilde{T}, \widetilde{R})$, this time by choosing $b = \coth(a/2)$, $c = 1/\sinh^2(a/2)$ and $d = \tanh(a/2)$ in equation (7). This gives the generating function

$$f(x) = -\coth(x/2) \ . \tag{67}$$

Then the transformation (6) takes the form

$$\widetilde{T} = \frac{\sinh\eta}{\cosh\chi - \cosh\eta} \quad , \quad \widetilde{R} = \frac{\sinh\chi}{\cosh\eta - \cosh\chi} \ . \tag{68}$$

If the universe model begins at $\eta = 0$, this transformation maps the region $0 < \eta < \chi$ onto the region $0 < \widetilde{T} < -1 - \widetilde{R}$, and the region $0 < \chi < \eta$ onto the region $0 < \widetilde{R} < -1 - \widetilde{T}$. On the other hand, if the universe is infinitely old, the region $-\chi < \eta < 0$ is mapped onto the region $1 + \widetilde{R} < \widetilde{T} < 0$, and the region $0 < \chi < -\eta$ onto the region $0 < \widetilde{R} < \widetilde{T} - 1$.

The inverse transformation is

$$\coth\eta = \frac{\left(\widetilde{R}^2 - \widetilde{T}^2\right) - 1}{2\widetilde{T}} \quad , \quad \coth\chi = \frac{\left(\widetilde{T}^2 - \widetilde{R}^2\right) - 1}{2\widetilde{R}} \ . \tag{69}$$

From equation (9) it follows that the line element again takes the form (58) with $(\widehat{T}, \widehat{R})$ replaced by $(\widetilde{T}, \widetilde{R})$. As drawn in the $(\widetilde{T}, \widetilde{R})$ spacetime diagram in Figure 7 and 8, both the world lines of points with $\chi = \chi_0$ and the simultaneity curves $\eta = \eta_0$ are hyperbolae, given respectively by

$$(\widetilde{R} + a_1)^2 - \widetilde{T}^2 = a_1^2 - 1 \quad \text{and} \quad (\widetilde{T} + b_1)^2 - \widetilde{R}^2 = b_1^2 - 1 \ , \tag{70}$$

where $a_1 = \coth\chi_0$ and $b_1 = \coth\eta_0$. Again these equations are valid both for universe models dominated by radiation and dust and by LIVE, the only difference being that $\eta_0 > 0$ with radiation and dust, and $\eta_0 < 0$ with LIVE. In the present case the line element is still given by equation (65).

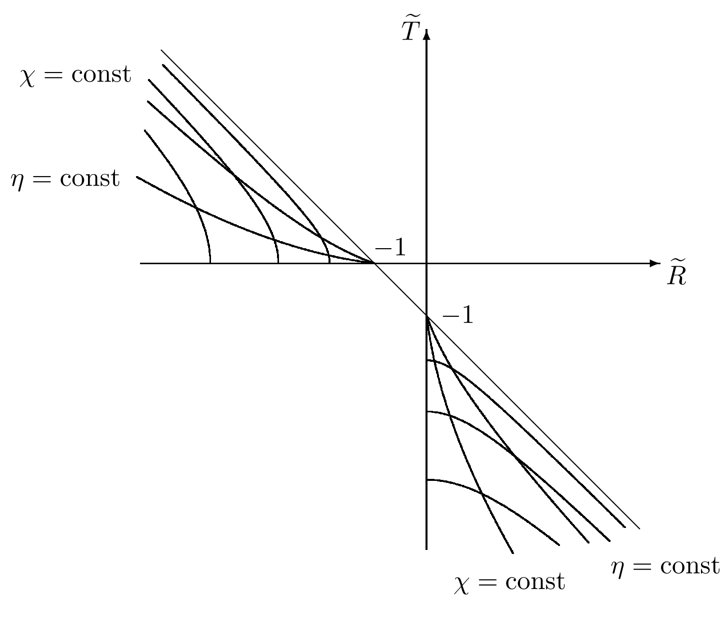

Figure 7. Minkowski diagram for universe models with negative spatial curvature containing dust and radiation with reference to the conformal coordinate system of type III defined in equation (68). The diagram shows world lines $\chi =$ constant and simultaneity curves $\eta =$ constant. The line $\widetilde{T} + \widetilde{R} + 1 = 0$ represents the limit $\chi \to \infty$, $\eta \to \infty$. Furthermore $\eta \to 0$ on the $\widetilde{R}$-axis, and $\eta \to \infty$ at the point $\widetilde{T} = -1$ on the $\widetilde{T}$-axis.

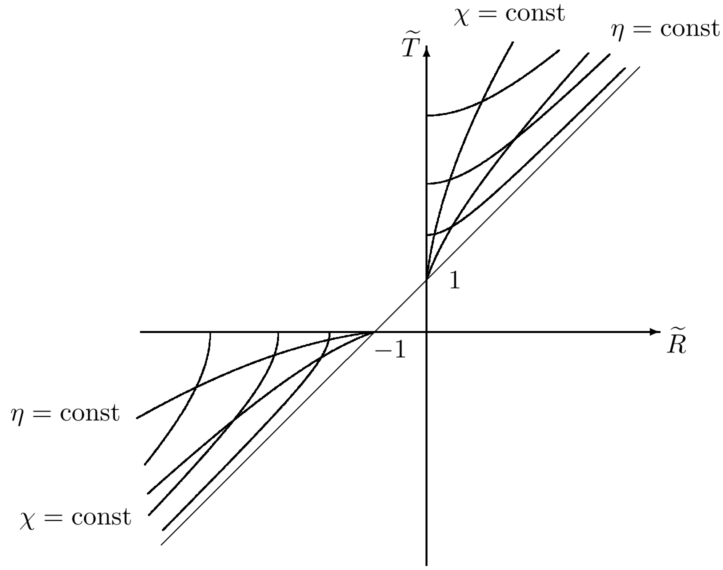

Figure 8. Minkowski diagram for universe models with negative spatial curvature containing vacuum energy with reference to the conformal coordinate system of type III defined in equation (68). The diagram shows world lines χ = constant and simultaneity curves η = constant. The line $\widetilde{T} = \widetilde{R} + 1$ represents the limit $\chi \to \infty$, $\eta \to -\infty$. Furthermore $\eta \to 0$ on the $\widetilde{R}$-axis, and $\eta \to -\infty$ at the point $\widetilde{T} = 1$ on the $\widetilde{T}$-axis.

9. Flat universe models

9.1. Conformal coordinates in flat universe models

In flat universe models η and χ already are conformal coordinates, corresponding to $a = 0$, $b = 0$, $c = 2$ and $d = 0$ in equation (7). This gives the generating function $f(x) = x$. However, $\eta =$ constant defines the same cosmic space as that given by $t =$ constant. But when $k = 0$, we can introduce a second type of conformal coordinates defined by $a = 1$, $b = 2$, $c = 2$ and $d = -1$ in equation (7). This gives the generating function

$$f(x) = -1/x \ . \tag{71}$$

The transformation (6) then takes the form

$$T = \frac{\eta}{\chi^2 - \eta^2} \quad \text{and} \quad R = \frac{\chi}{\eta^2 - \chi^2} \ . \tag{72}$$

If the universe model begins at $\eta = 0$, this transformation maps the region $0 < \eta < \chi$ onto the region $0 < T < -R$, and the region $0 < \chi < \eta$ onto the region $0 < R < -T$. On the other hand, if the universe is infinitely old, the region $-\chi < \eta < 0$ is mapped onto the region $R < T < 0$, and the region $0 < \chi < -\eta$ onto the region $0 < R < T$. The inverse transformation has the same form,

$$\eta = \frac{T}{R^2 - T^2} \quad \text{and} \quad \chi = \frac{R}{T^2 - R^2} \ . \tag{73}$$

Since $\partial T/\partial \eta > 0$, T increases in the same direction as η and t.

The line element takes the form

$$ds^2 = \left[\frac{a(\eta(T,R))}{T^2 - R^2}\right]^2 (-dT^2 + dR^2 + R^2 d\Omega^2) \ . \tag{74}$$

From equation (46) in reference [1] it follows that a particle with χ = constant has a recession velocity in the (T, R)-system given by

$$V = \frac{2TR}{T^2 + R^2} \ , \tag{75}$$

Again the initial recession velocity vanishes. Furthermore $V > 0$ for $TR > 0$ and $V < 0$ for $TR < 0$. Note that $V < 0$ corresponds to expansion and $V > 0$ to contraction when $R < 0$.

A flat universe model with radiation and dust has [2]

$$a = \frac{1}{2}\alpha\eta^2 + \beta\eta \quad , \quad t = \frac{1}{6}\alpha\eta^3 + \frac{1}{2}\beta\eta^2 \tag{76}$$

where

$$\alpha = \Omega_{m0}/2 \quad \text{and} \quad \beta = \sqrt{\Omega_{\gamma 0}} \ . \tag{77}$$

The relationship between the conformal time at $R = 0$ and the cosmic time is

$$t = \frac{3\beta T - \alpha}{6T^3} \ . \tag{78}$$

The conformal clocks go faster than the cosmic ones. Here $\eta \in <0, \infty>$ when $t \in <0, \infty>$. From equation (72) it then follows that the conformal time T is negative for these universe models, and $T \to 0$ when χ is constant and $\eta \to \infty$. The line element as expressed by the (T, R)-coordinates takes the form

$$ds^2 = \left[\frac{(\Omega_{m0}T - 4\sqrt{\Omega_{\gamma 0}}\,(T^2 - R^2))\,T}{4(T^2 - R^2)^3}\right]^2 ds_M^2 \ . \tag{79}$$

The Einstein-deSitter universe is a dust dominated, flat universe. It has $\beta = \Omega_\gamma = 0$. The line element for this universe in the T, R-coordinate system is

$$ds^2 = \left[\frac{\Omega_{m0}T^2}{4(T^2 - R^2)^3}\right]^2 ds_M^2 \ . \tag{80}$$

The world lines of the reference particles in the cosmic coordinate system, $\chi = \chi_0$, in the conformal reference frame are given by

$$(R + a_2)^2 - T^2 = a_2^2 \ , \tag{81}$$

where $a_2 = (2\,\chi_0)^{-1} > 0$. The simultaneity curves of the cosmic space, $\eta = \eta_0$, as given in the conformal system, are

$$(T + b_2)^2 - R^2 = b_2^2 \ , \tag{82}$$

where $b_2 = (2\eta_0)^{-1}$. These two sets of hyperbolae are drawn for a universe model with matter and radiation in Figure 9.

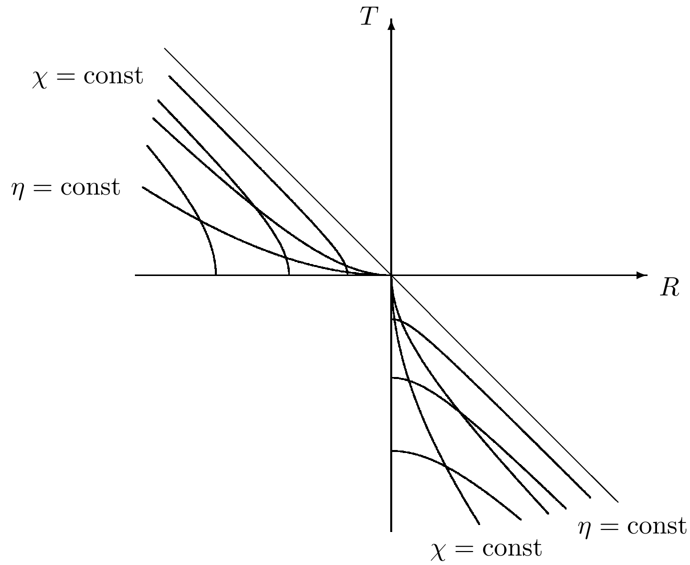

Figure 9. Minkowski diagram with reference to the conformal coordinate system for a flat universe model dominated by matter and radiation. The diagram shows world lines χ = constant and simultaneity curves η = constant. For this universe model the Hubble flow is expanding for $T > 0$ and contracting for $T < 0$ relative to the conformal system.

The Minkowski diagram in Figure 9 shows that the universe with radiation and dust appears at $T = 0$ having an infinitely great extension. Then a hole develops expanding with the velocity of light. For an infinitely large conformal time the Hubble flow expands relative to the conformal frame. As T approaches infinity, the conformal clocks are reset to come from minus infinity, and R changes from negative to positive values. The universe then contracts relative to the conformal frame to vanishing extension at $T = 0$. Hence during the period with positive time the conformal frame contracts relative to the Hubble flow, and during the succeeding period with negative time the conformal frame expands faster than the Hubble flow.

The world lines of the reference particles of the conformal coordinate system, $R = R_0$, in the (η, χ)-system are given by (see Figure 10)

$$(\chi + a_3)^2 - \eta^2 = a_3^2 \ , \tag{83}$$

where $a_3 = (2R_0)^{-1}$. The simultaneity curves of the conformal space, $T = T_0$, have the equation

$$(\eta + b_3)^2 - \chi^2 = b_3^2 \ , \tag{84}$$

where $b_3 = (2T_0)^{-1}$.

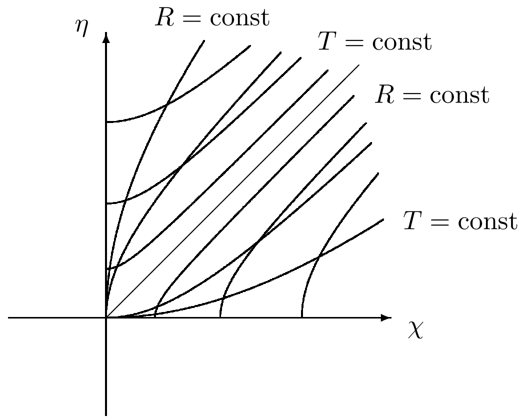

Figure 10. Minkowski diagram with reference to the (η, χ)-system for a flat universe model dominated by matter and radiation. The diagram shows world lines R = constant and simultaneity curves T = constant. In this universe model the conformal system expands relative to the Hubble flow.

The world lines of the reference particles with R = constant in the conformal coordinate system is given by equation (36) in reference [1] which leads to

$$\left(\frac{d\chi}{d\eta}\right)_{R=R_1} = -\tanh\theta = \frac{2\eta\chi}{\eta^2+\chi^2} > 0 \ . \tag{85}$$

This means that the (T, R)-system expands relative to the (η, χ)-system. Hence the (η, χ)-system contracts relative to the (T, R)-system with a velocity

$$\left(\frac{dR}{dT}\right)_{\chi=\chi_1} = \tanh\theta = \frac{2TR}{T^2+R^2} < 0 \tag{86}$$

as shown in Figure 9.

In a flat LIVE dominated universe the scale factor is

$$a(t) = e^{\widehat{H}_\Lambda (t-t_0)} \ , \tag{87}$$

where $\widehat{H}_\Lambda$ is given in equation (47) and is constant. Then the parametric time η is

$$\eta = -\frac{1}{\widehat{H}_\Lambda} \, e^{-\widehat{H}_\Lambda (t-t_0)} \ , \tag{88}$$

which increases from $-\infty$ to 0 as t increases from $-\infty$ to ∞. The world lines χ = constant and the simultaneity curves η = constant are shown in a Minkowski diagram with reference to the conformal coordinate system for this universe model in Figure 11. The corresponding world lines R = constant and simultaneity curves T = constant are shown in Figure 12.

In terms of the parametric time η or of the conformal coordinates R and T, the scale factor may be written

$$a(\eta) = -\frac{1}{\widehat{H}_\Lambda \eta} = \frac{T^2-R^2}{\widehat{H}_\Lambda T} \ . \tag{89}$$

In this case the line element in conformal coordinates takes the form [14]

$$ds^2 = \frac{1}{\widehat{H}_\Lambda^2 T^2} ds_M^2 \tag{90}$$

as in the negatively curved case. There is no continual creation in such a universe model.

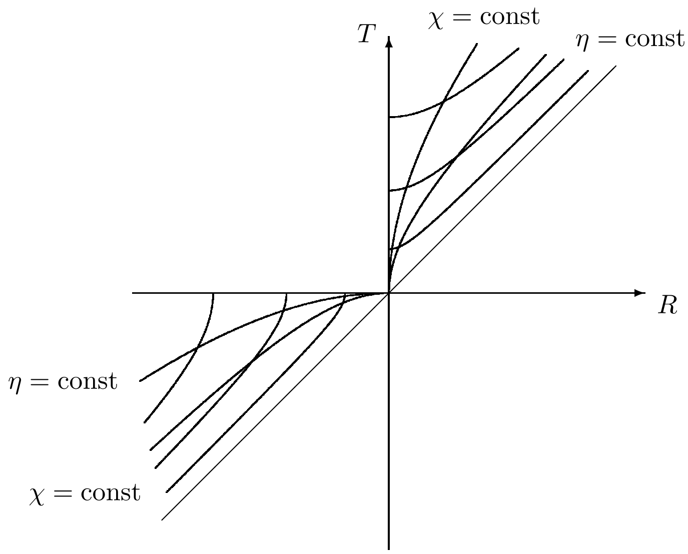

Figure 11. Minkowski diagram with reference to the conformal coordinate system for a flat universe model dominated by vacuum energy. The diagram shows world lines $\chi =$ constant and simultaneity curves $\eta =$ constant. For this universe model the Hubble flow is expanding for $T > 0$ and contracting for $T < 0$ relative to the conformal system.

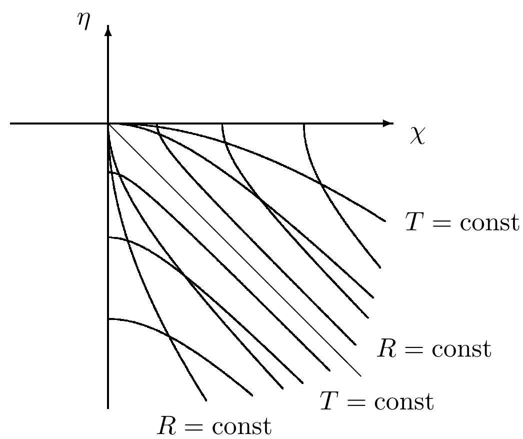

Figure 12. Minkowski diagram with reference to the (η, χ)-system for a flat universe model dominated by vacuum energy. The diagram shows world lines $R =$ constant and simultaneity curves $T =$ constant. In this universe model the conformal system contracts relative to the Hubble flow.

The evolution of the flat LIVE dominated universe in the CFS-system is as follows. The universe starts at $T = 0$ and expands from an initial state with vanishing extension. In this era the conformal space has finite extension, although it expands without limit. Again, as the conformal time T approaches infinity, the clocks are reset to come from minus infinity, and the sign of the radial coordinate is changed. The conformal space then has infinite extension and is contracting. The contraction slows down to a final state at rest at $T = 0$, in which the conformal space still has infinite extension.

9.2. Particle horizon in flat universe models using conformal time

We shall here discuss the particle horizon in flat universe models from the perspective of the conformal coordinates T and R. Let us first consider a universe model dominated by vacuum energy. It extends backwards in time to $t \to -\infty$. In such a universe model there is no particle horizon. However, this universe model is not realistic. We cannot expect that the general theory of relativity can give a realistic description of spacetime before the Planck time. Hence we assume that the inflationary era, which may be described classically, starts at the Planck time η_{Pl}. Then there is a particle horizon around an observer at P with coordinates $T = T_0 > 0$ and $R = 0$. As in section 4.3 the horizon H is defined as the intersection between the past light cone at P, $R = T_0 - T$, and the space at $\eta = \eta_{Pl}$. In the present case this space is represented by the hyperbola $R^2 = T^2 + 2b_{Pl}T$ where $b_{Pl} = (2\eta_{Pl})^{-1}$. This gives

$$T_H = \frac{T_0^2}{2(T_0 + b_{Pl})} \ , \quad R_H = \frac{(T_0 + 2b_{Pl})T_0}{2(T_0 + b_{Pl})} \ . \tag{91}$$

Inserting these expressions into the transformation (73) gives

$$\chi_H = -\frac{1}{T_0} - \frac{1}{2b_{Pl}} = -\frac{1}{T_0} - \eta_{Pl} \ . \tag{92}$$

Since P has coordinates $T = T_0$ and $R = 0$, the transformation (73) also gives

$$\eta_P = -\frac{1}{T_0} \ . \tag{93}$$

Hence the equation of the particle horizon for a flat universe model starting at the Planck time is

$$\chi_H = \eta_P - \eta_{Pl} \ . \tag{94}$$

This has here been deduced by using the spacetime diagram in Figure 11 with respect to the conformal coordinates T and R. Equation (94) is in accordance with equation (10) in reference [1] with $t_i = t_{Pl}$.

10. The CFS Hubble parameters of some universe models

The behaviour of the conformal space in different universe models may be investigated by calculating the CFS Hubble parameter defined in equation (48) in reference [1]. We first consider negatively curved universe models with respectively dust and radiation as described in CFS systems of type I. Using equation (35) we obtain

$$H_R = \frac{2T_i T}{(\sqrt{T^2 - R^2} - T_i)^3} \tag{95}$$

for a dust dominated universe, and by means of equation (36) we get

$$H_R = \frac{2T_i^2 T}{(T^2 - R^2 - T_i^2)^2} \tag{96}$$

for a radiation dominated universe. As shown in section 4.4 there is continual creation at the boundary of the conformal space given by $T^2 - R^2 = T_i^2$. Hence the CFS Hubble parameter for both of these universe models approaches infinity at the boundary with continual creation.

The CFS Hubble parameter for a negatively curved radiation dominated universe with conformal coordinates of type II and III is

$$H_R = \mathrm{sgn}(T)\, \frac{(1+\widehat{T}^2-\widehat{R}^2)^2-4\widehat{T}^2(\widehat{T}^2-\widehat{R}^2)}{4\beta\widehat{T}^2}\ . \tag{97}$$

Correspondingly for a flat universe one obtains

$$H_R = \frac{|T^2-R^2|\,(3T^2+R^2)}{\beta T^2}\ . \tag{98}$$

The Hubble parameter for a LIVE dominated universe model with $k = -1$ and CFS coordinates of type I is

$$H_R = \sqrt{\Omega_0}\,\widehat{H}_0\,\frac{T}{T_f}\ . \tag{99}$$

In this case the conformal Hubble parameter in independent of the position R. The Hubble parameter for the same universe model but with CFS coordinates of type II and III is

$$H_R = -\widehat{H}_\Lambda\,\mathrm{sgn}(T) \tag{100}$$

where we have omitted the hat and the tilde on T. This formula is also valid for a flat LIVE dominated universe. The conformal space therefore expands for $T < 0$ and contracts for $T > 0$. This behaviour is a result of two competing motions. The Hubble flow expands exponentially. But as seen in Figure 12, the conformal reference frame contracts relative to the Hubble flow. The sign of the conformal Hubble parameter shows that the expansion dominates for $T < 0$ corresponding to the region $-\chi < \eta < 0$ in the (η, χ)-plane, while the contaction of the conformal system dominates for $T > 0$ corresponding to the region $0 < \chi < -\eta$.

11. Conclusion

In the (η, χ)-system, time and space are created at the Big Bang singularity at $t = 0$. In the (T, R)-system, new space is also created at later times. Big Bang happens continually at the boundary of space, i.e. on the hyperbola $T^2 - R^2 = T_i^2$ in Figure 1. Just as our universe is said not to exist before $t = \eta = 0$, our universe did not exist below this hyperbola. Or, alternatively if cosmic time and space was created at the Planck time, conformal space is continually created at the Planck boundary.

What then about continual creation of new space? Is this only a coordinate effect? In a way it is. As made clear in the introduction conformal space is a coordinate space. But cosmic space is also a coordinate space. The first one is defined by $T =$ constant and

the second one by $t =$ constant. Is the one more fundamental than the other?

A difference between the cosmic and the conformal space is that the reference particles of the first one are freely moving, and those of the second ones must be acted upon by non-gravitational forces. The reference particles of the cosmic space constitute an inertial flow, while those of the conformal space do not. In this sense the cosmic space is more fundamental than the conformal space.

Continual creation of conformal space, matter and energy is physical and real. But it belongs to a CFS picture of the universe which does not have the same physical significance as the picture of our universe based upon cosmic time and freely moving reference particles, because these particles which define the Hubble flow, are in fact the clusters of galaxes.

In the present paper we have described FRW universe models with negative and vanishing spatial curvature using different types of conformally flat spacetime coordinates. They are comoving in reference frames that move relative to each other. Due to the relativity of simultaneity they therefore define different spatial sections of spacetime. Hence the picture that they give of the creation and the evolution of the universe are different, and also rather strange when we are used to the standard picture of Big Bang happening everywhere at a certain moment, $t = 0$, provided by using cosmic coordinate time and spatial coordinates comoving with free particles.

In the picture associated with the first type of CFS coordinates the universe first appears at the spatial origin of the coordinate system and then expands with superluminal velocity. The universe is inhomogeneous, and the energy density and temperature increase towards infinity at the expanding front which defines the boundary of the universe. New space, matter and energy are created at this front. Inside the front space expands with a constant, subluminal velocity. This is not a spaces defined by free particles, but by the reference particles of the CFS coordinate system.

The motion of the CFS reference particles relative to those of the standard cosmic coordinate system, defining the inertial Hubble flow, has been depicted for all three types of CFS coordinates.

The picture of the universe with reference to CFS coordinates of the second type depends upon the contents of the universe. The matter and radiation dominated universe contracts and has finite age and spatial extension, while the corresponding universe dominated by vacuum energy expands.

The CFS picture of type III of the evolution of a universe with radiation and dust is rather pathological. Looking at Figure 7, one might expect that the universe starts at an infinitely past conformal time. This is, however, no so. A dust and radiation dominated universe starts at a cosmic time $t = 0$. From equation (31) it follows that this corresponds to a parametric time $\eta = 0$. Equation (68) therefore implies that the universe starts at a conformal time $\widetilde{T} = 0$. Figure 7 shows that the universe then starts with a hole in a finite region around $\widetilde{R} = 0$ where the conformal space does not exist, surrounded by a region with radiation and dust of infinitely great extension. The conformal space then expands and so does the hole. As $\widetilde{T}$ approaches infinity, the clocks are reset to come from minus infinity, and $\widetilde{R}$ changes from negative to positive values. A negative sign of $\widetilde{R}$ can be removed by changing the coordinates θ and ϕ corresponding to a reflection through the origin in space. This means that the expansion is replaced by a contraction. The

conformal space then contracts to a vanishing extension at $\widetilde{T} = -1$.

The corresponding picture for a LIVE dominated universe with negative spatial curvature is as follows. Equation (46) implies that such a universe start at a cosmic time $t = 0$. From equation (49) it follows that this corresponds to a parametric time $\eta \to -\infty$. Equation (68) therefore implies that this universe starts at a conformal time $\widetilde{T} = 1$. Figure 8 now shows that the conformal space then appears with vanishing extension and expands towards infinite extension as the conformal time approaches infinity. At this point the clocks are reset to come from minus infinity, and $\widetilde{R}$ changes from positive to negative values, so that the expansion is replaced by a contraction. The conformal space then contracts to a final state at $\widetilde{T} = 0$ similar to the initial state with radiation and dust.

The evolution of the conformally flat space of a universe with critical density dominated by dust and radiation, corresponding to the evolution using cosmic time, is rather peculiar. The Big Bang is replaced by the appearance of a hole in space which expands with a velocity of light with center at the observer, even if the space as defined by constant cosmic time is homogeneous, and the observer has an arbitrary position. Then, at an infinitely far future, the CFS clocks are reset to minus infinity, and the universe contracts relative to the conformal frame.

To the CFS observer the resetting of the clocks may seem rather artificial. It would therefore be tempting to reinterpret Figure 9 saying that the universe started by contracting from an infinitely remote past. Such a reinterpretation is however not permitted. We know from the usual description of the universe in terms of cosmic time that objects in the universe grow older with increasing cosmic time. Hence obsevations would show that at $T < 0$ an object in the universe would be older than it was at $T > 0$.

By applying the rule in Appendix B in reference [1] for composing generating functions one may obtain a deeper understanding of the conformal coordinate transformations and find new ones. Note the similarity between the transformations to CFS coordinates of type II and III in a universe models with negative spatial curvature given by the equations (56) and (68). One may wonder whether this points to a relationship between these trasnformations. The answer is that the generating function in (67) may be obtained as a composition of the generating function (55) for the CFS coordinates of type II in a universe with negative spatial curvature and the generating function (71) for the CFS coordinates of type II in a flat universe.

It remains to describe universe models with positive spatial curvature in terms of CFS coordinates. This will be done in the third and final paper in this series, where we will also consider Penrose diagrams and compositions of transformations between CFS coordinates for universe models with different spatial curvature.

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